Odd disjoint trails and totally odd graph immersions

Ross Churchley

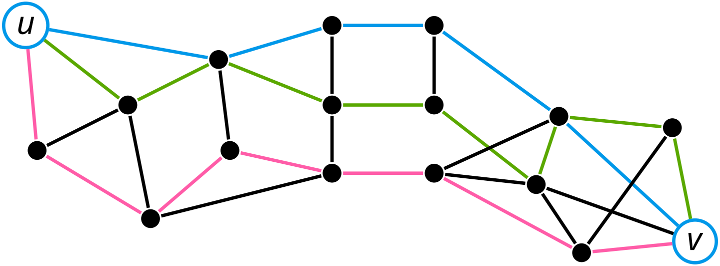
PhD defence, 10 January 2017

Simon Fraser University

1. Packing odd trails

Theorem (Menger)

The maximum number of edge-disjoint (u, v)-paths in a graph is the minimum size of a (u, v)-cut.



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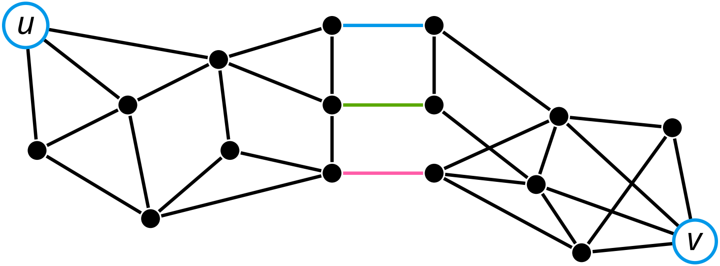

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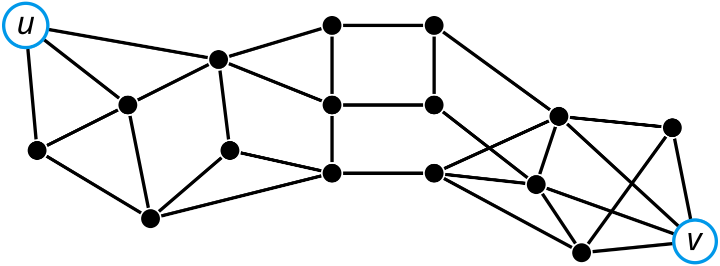
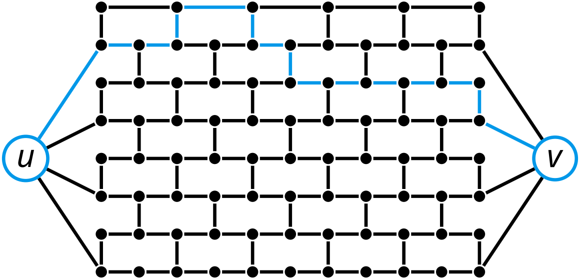


Question
How many edge-disjoint odd (u, v)-trails are there?

Aside: why trails?



A graph can have very few disjoint odd (u, v)-paths

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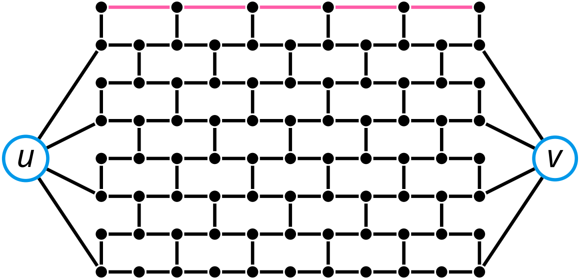
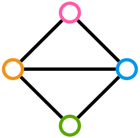
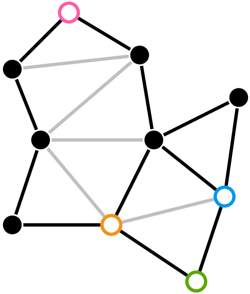


A graph can have very few disjoint odd (u, v)-paths even if many edges are needed to cover all such paths.

2. Totally odd immersions

An immersion of H in G maps vertices to vertices and edges to edge-disjoint trails.





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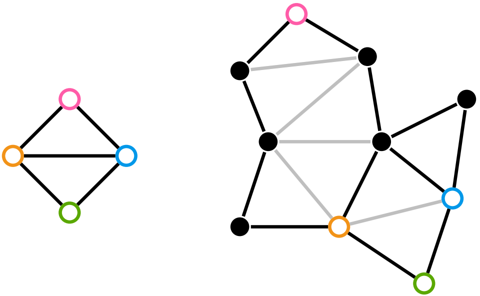
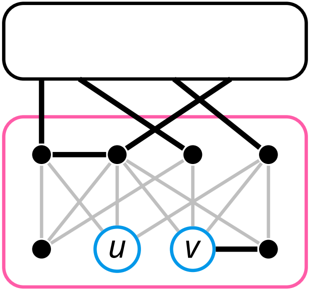


Question

When does a graph have an immersion of H where all the trails have odd length?

Main contributions



The **perimeter** of a vertex-set X wrt bipartite subgraph H:

$$p(X, H) = |E(X) \setminus E(H)| + \frac{1}{2}|\delta(X)|$$

1. Bounds for the “**odd edge-connectivity**”.
2. Better bounds for **Eulerian** graphs.
3. A **submodular** inequality for perimeter.
4. A Gomory–Hu Theorem for minimum-perimeter sets.
5. A **rough structure theorem** for graphs with no totally odd K_t-immersion.
6. More results forcing totally odd immersions.

Packing edge-disjoint (u, v)-trails of odd length

Theorem
Given G, vertices u, v, and H a maximum bipartite subgraph, if $G + uv$ is 2-edge-connected, then either:

1. *G has k edge-disjoint odd (u, v)-trails;*

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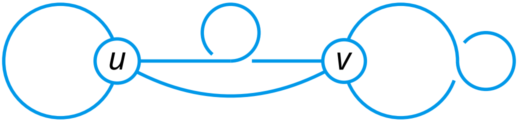
3. *u, v are on the same side of H, and G has a set R with $u, v \in R$ and perimeter $p(R, H) \leq 3(k - 1)$.*

Proof

Trick: first look for odd trails with ends in $\{u, v\}$.



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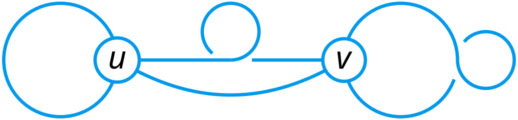

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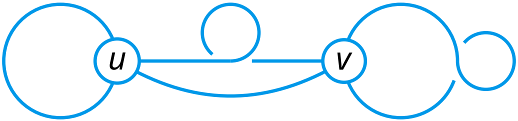
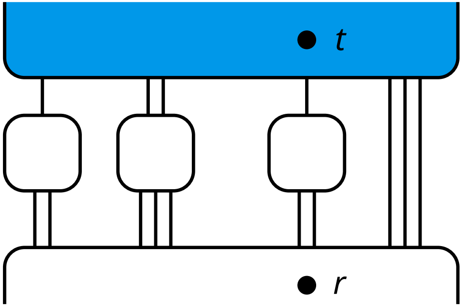


In fact, find trails with exactly one edge outside H greedily choosing these edges.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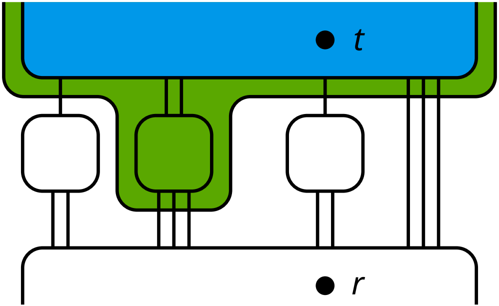
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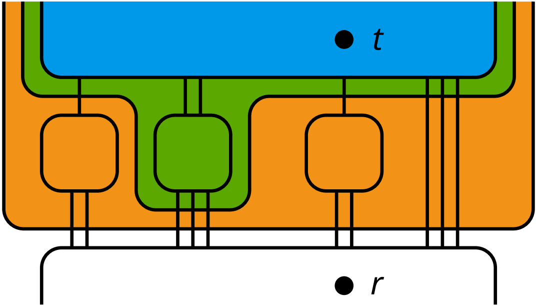
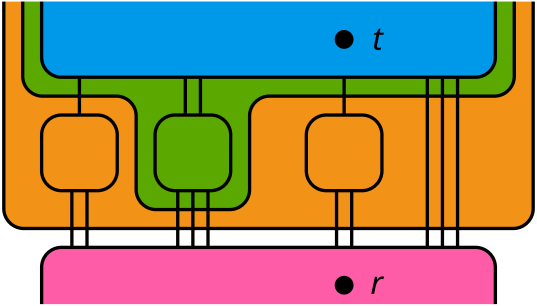
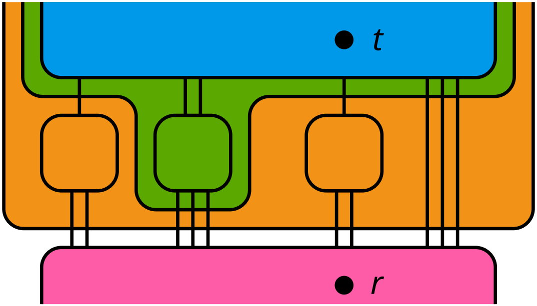


The structure of an auxiliary graph gives a small cut. . .



The structure of an auxiliary graph gives a small cut…
and a set with $p(R, H) \leq 3k$ in G.



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We also have k trails, which we can turn into
(u, v)-trails given sufficient edge-connectivity.

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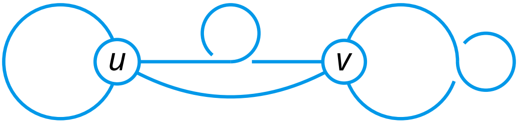
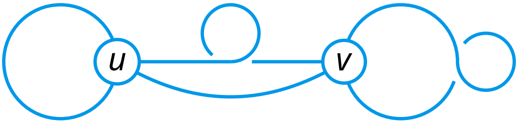
Corollary

A graph has either k edge-disjoint odd (u, v)-trails or a set of at most $6k - 2$ edges intersecting all such trails.

Packing edge-disjoint odd (u, v)-trails in Eulerian graphs



u
v



There's another way to find odd trails with ends in $\{u, v\}$, and in Eulerian graphs, its bound relates to perimeter.

Theorem (Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour)
Let H be a group-labelled graph and let $A \subseteq V(H)$. The max # of vertex-disjoint non-zero A-paths in H is

$$\min_{S,D} \left\{ |S| + \sum_{\substack{\text{components } K \\ \text{of } H-S-D}} \left\lfloor \frac{|(A \cup V(D)) \cap V(K)|}{2} \right\rfloor \right\}$$

where $S \subseteq V(H)$ and $D \subseteq E(H)$ has no non-zero cycles and no non-zero A-paths.

Theorem
The max # of edge-disjoint odd (r, r)-trails in a graph G is

$$\min_{R,H} \left\{ |E(R) \setminus E(H)| + \sum_{\substack{\text{components } K \\ \text{of } G-R}} \left\lfloor \frac{|\delta(V(K))|}{2} \right\rfloor \right\}$$

where $r \in R \subseteq V(G)$ and $H \subseteq G$ is bipartite.

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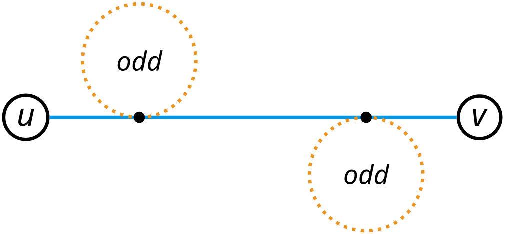
If G is Eulerian, this is a minimum perimeter!

Modify a collection of $\{u, v\}$-trails to maximize the number of edges used.



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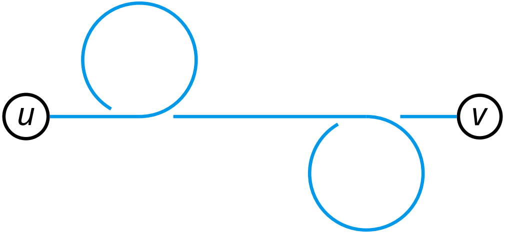
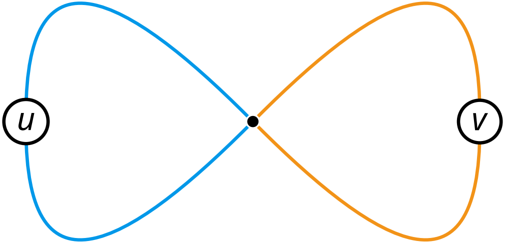


Then rearrange intersecting odd trails into (u, v)-trails.



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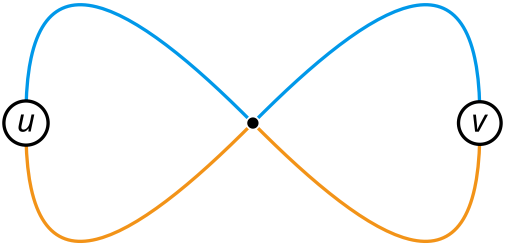
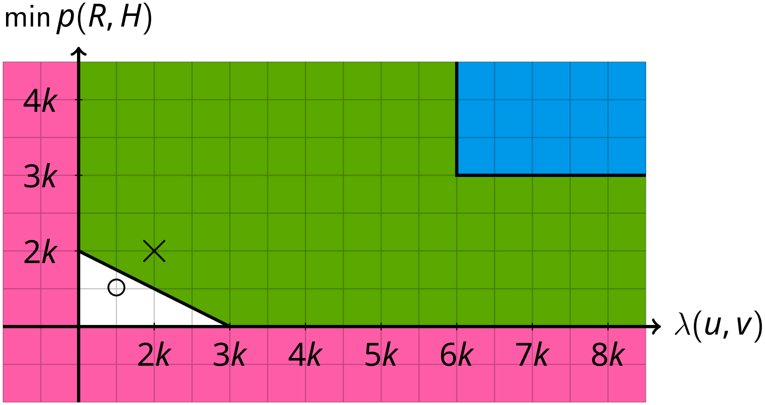


Theorem
Let $\alpha \in (0, 1]$. *If G is Eulerian, either*

1. *G has k edge-disjoint odd* (u, v)*-trails;*
2. *G has a* (u, v)*-cut with fewer than* $(3 - 2\alpha)k$ *edges; or*
3. *G has a bipartite subgraph H with u, v on the same side and a set R with* $u, v \in R$ *and* $p(R, H) < (1 + \alpha)k$.

Summary of our packing results



No *k* odd trails

k odd trails if Eulerian

k odd trails

Perimeter and submodularity

A function $f : 2^X \to \mathbb{R}$ is called submodular if

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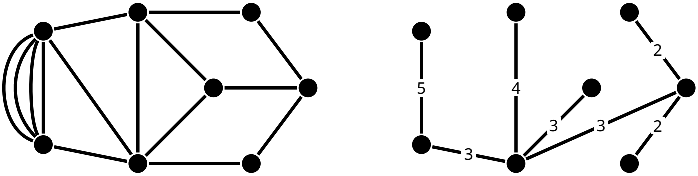
For example, $|\delta X|$ is submodular.

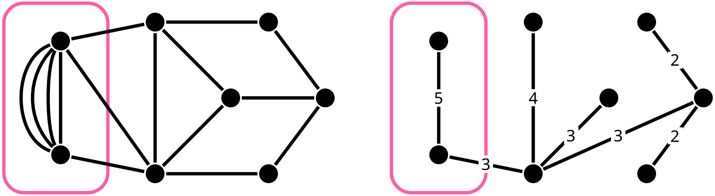
Theorem
For a fixed subgraph H, the perimeter of a set X wrt H

$$p(X, H) = |E(X) \setminus E(H)| + \frac{1}{2}|\delta(X)|$$

is a submodular function.



Theorem (Gomory/Hu)
For every graph there is an edge-weighted tree which encodes minimum cuts for each vertex pair.



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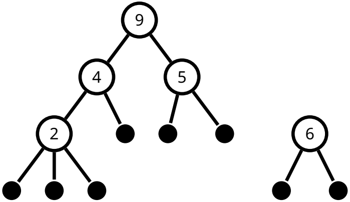
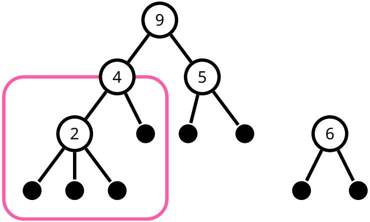
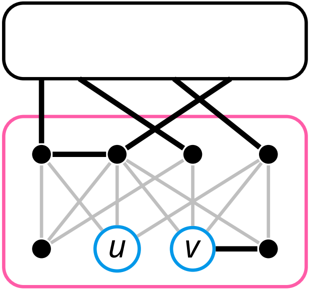


Theorem
For every graph there is an vertex-weighted rooted forest encoding minimum-perimeter sets for each vertex.





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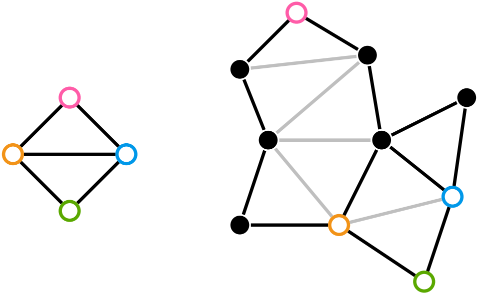
For every graph there is an vertex-weighted rooted forest encoding minimum-perimeter sets for each vertex.

Totally odd immersions of cliques

An immersion of H in G maps vertices to vertices and edges to edge-disjoint trails.



An immersion of K_t is a set of t vertices in G and an edge-disjoint collection of $\binom{t}{2}$ trails connecting the pairs.

Theorem (DeVos, McDonald, Mohar, Scheide; Wollan)
If G has no K_t-immersion, it has a laminar family of cuts of size $< (t-1)^2$ which partition $V(G)$ into sets of size $< t$.

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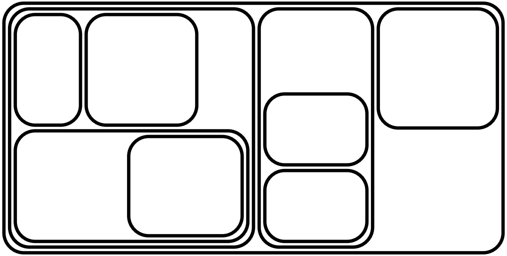

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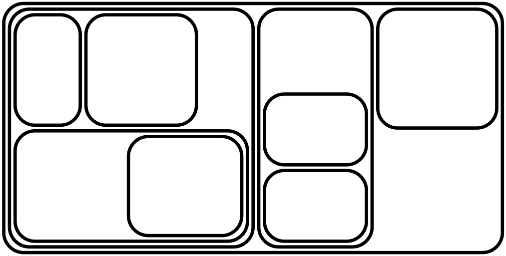

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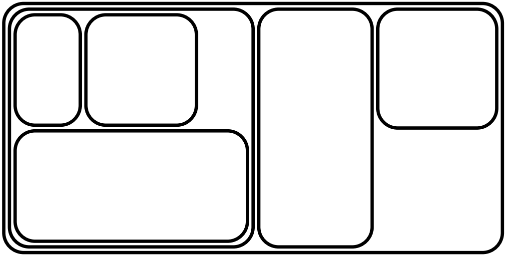

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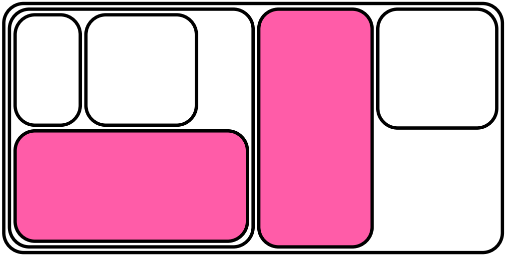

More totally odd immersions

Conjecture (Lescure, Meyniel; Abu-Khzam, Langston)
Every graph with chromatic number t has a K_t-immersion.

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Every graph with chromatic number t has a K_t-immersion.

Question
Does every graph with chromatic number t have a totally-odd K_t-immersion?

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Almost every random graph has a totally-odd (strong) immersion of the complete graph on $O(p^{3/2}n)$ vertices.

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Corollary
The "totally-odd immersion Hadwiger Conjecture" is true for almost all graphs.

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Question
Do graphs in $\mathcal{G}_{n,1/2}$ have totally-odd immersions of $K_{n/2}$?

Theorem

Let G be a graph with $\alpha(G) = 2$ and $\delta(G) \geq \frac{3}{4}n$.

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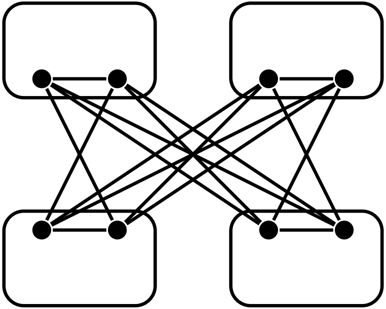
Theorem

Let G be a graph with $\alpha(G) = 2$ and $\delta(G) \geq \frac{3}{4}n$. Then G has a totally-odd immersion of $K_{n/2}$ rooted on any chosen rainbow set of vertices.



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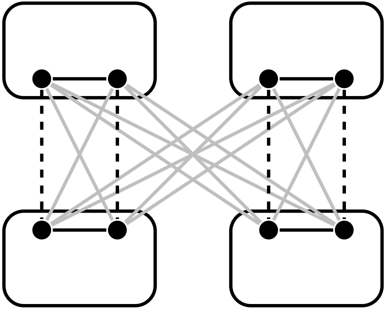

Conclusion

1. Introduced the **perimeter** measure.
2. Found bounds for the “**odd edge-connectivity**”.
3. Explored the **submodular** inequality for perimeter.
4. Proved conditions forcing **totally-odd immersions**.
5. Described polytime **algorithms**.

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