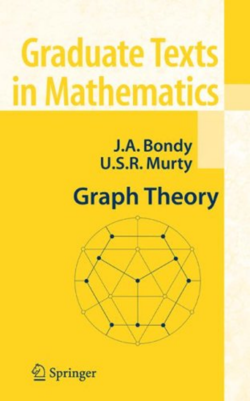
Graph theory haiku

Three short and beautiful proofs

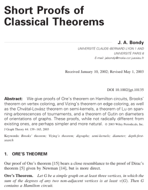
Ross Churchley

University of Victoria

December 21, 2021



Adrian Bondy



Short Proofs of Classical Theorems

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Abstract: We give proofs of Ore's theorem on Hamilton circuits, Brooks' theorem on vertex coloring, and Vizing's theorem on edge coloring, as well as the Chvátal-Lovász theorem on semi-kernels, a theorem of Lu on spanning arborescences of tournaments, and a theorem of Gutin on diameters of orientations of graphs. These proofs, while not radically different from existing ones, are perhaps simpler and more natural.

J Graph Theory 44: 159–165, 2003

Keywords: *Brooks' theorem; Vizing's theorem; digraphs; semi-kernels; diameter; depth-first-search*

1. ORE'S THEOREM

Our proof of Ore's theorem [15] bears a close resemblance to the proof of Dirac's theorem [5] given by Newman [14], but is more direct.

Ore's Theorem. *Let G be a simple graph on at least three vertices, in which the sum of the degrees of any two non-adjacent vertices is at least $v(G)$. Then G contains a Hamilton circuit.*

- **Ore's theorem** (1960) on Hamilton circuits
- **Brooks' theorem** (1941) on vertex colouring
- **Vizing's theorem** (1964) on edge colouring
- The Chvátal-Lovász theorem (1974) on semi-kernels
- Lu's theorem (1996) on arborescences of tournaments
- Gutin's theorem (1994) on diameters of graph orientations

Ore's Theorem

Let G be a simple graph on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for any nonadjacent u, v. Then G contains a Hamilton cycle.

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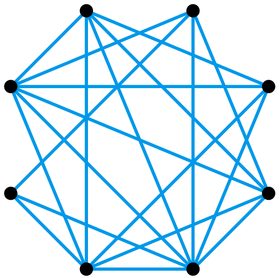
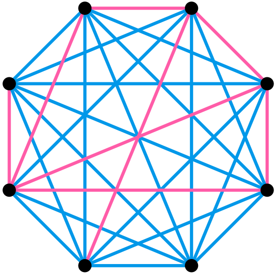
Colour G blue



Let G be a simple graph on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for any nonadjacent u, v. Then G contains a Hamilton cycle.

Colour G blue and add red edges to fill out K_n.



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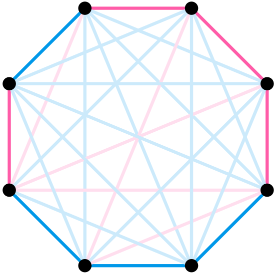
Colour G blue and add red edges to fill out K_n. Pick a Hamilton cycle C.



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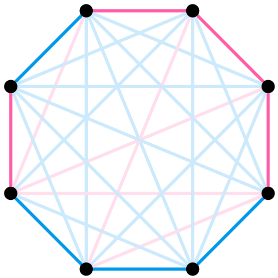

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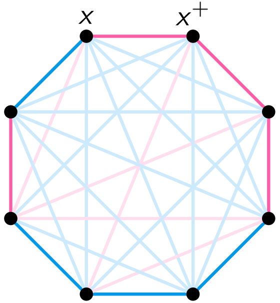

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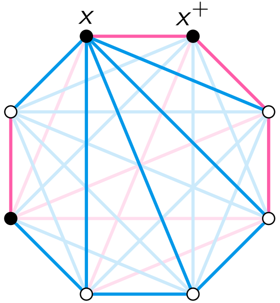

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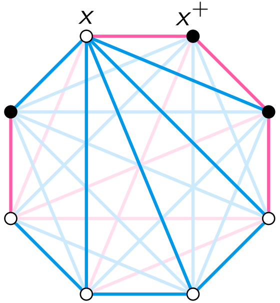

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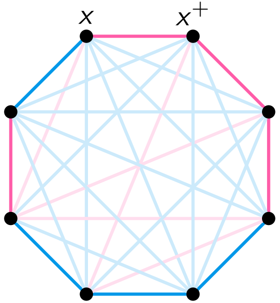
$$d_G(x^+) \geq n - d_G(x)$$



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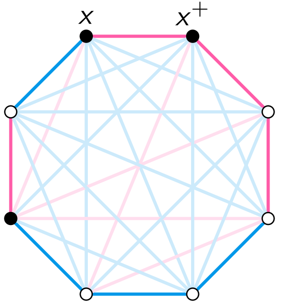

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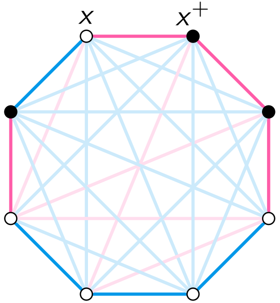

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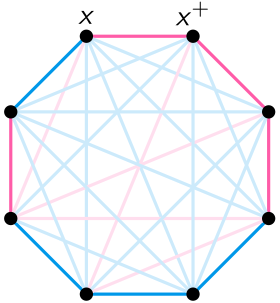

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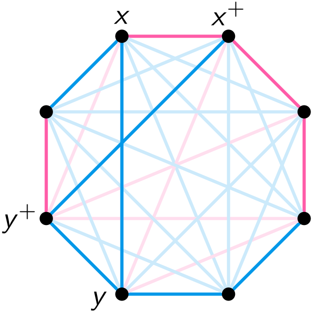

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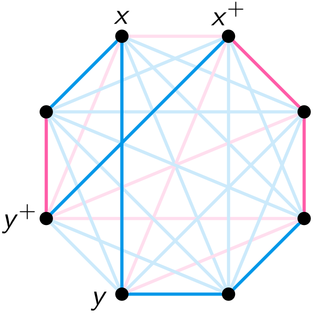
So x^+ is adjacent to a $y^+ \in S^+$ and we can get a "bluer" cycle.



□

A red-blue K_n:
bluest Hamilton circuit
lies fully in G.

Brooks' Theorem

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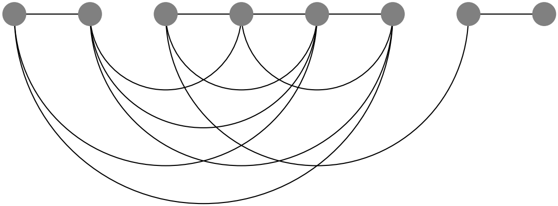
A greedy colouring always has at most $\Delta(G) + 1$ colours:



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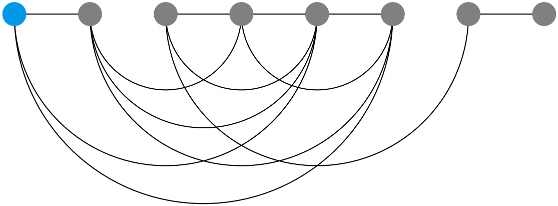

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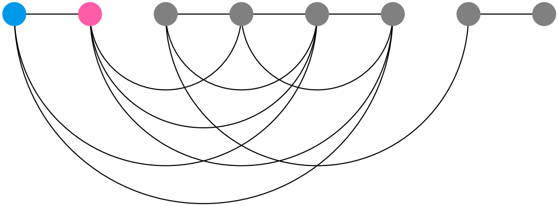

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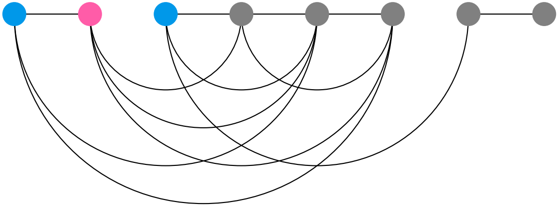

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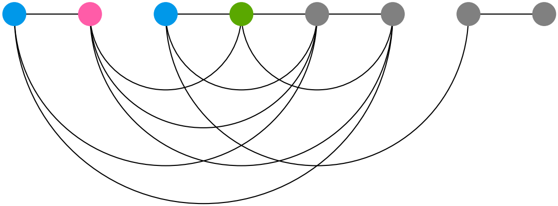

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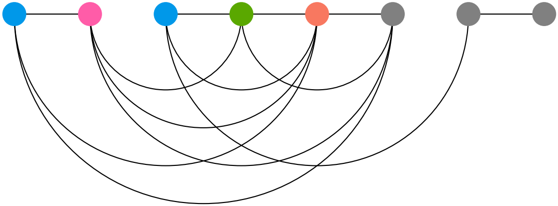

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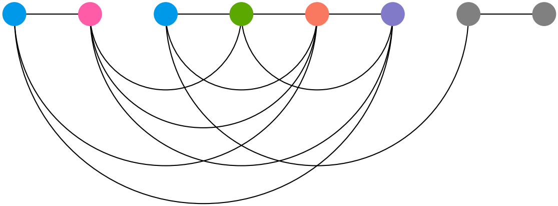

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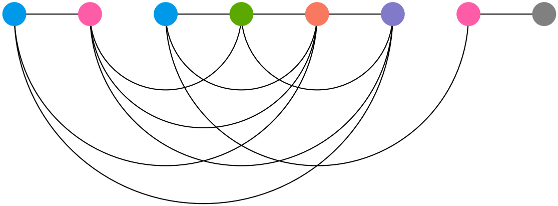

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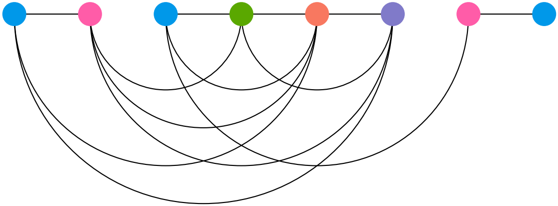

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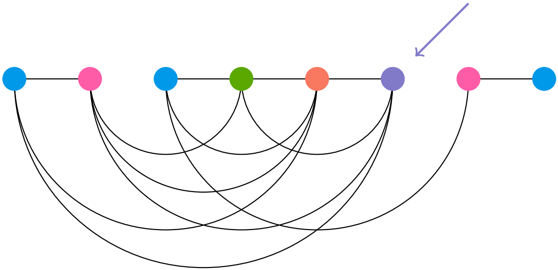

How can we avoid using the last colour? Maybe we can order the vertices such that no vertex has $\Delta(G)$ neighbours before it.

If G is connected and is not an odd cycle or a clique, then $\chi(G) \leq \Delta(G)$.

Case 1: G is not regular.

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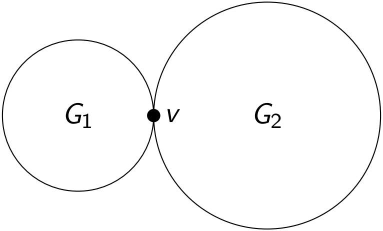
A greedy colouring according to this order uses at most $\Delta(G)$ colours.

If G is connected and is not an odd cycle or a clique, then $\chi(G) \leq \Delta(G)$.

Case 2: G is regular, but it has a cut vertex v.

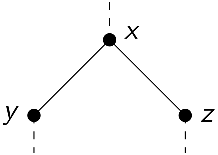
Then we can split G up into two graphs G_1, G_2 which are not regular.



By Case 1, we can colour G_1 and G_2 with $\Delta(G)$ colours which agree on v. This gives us a colouring of G.

If G is connected and is not an odd cycle or a clique, then $\chi(G) \leq \Delta(G)$.

Case 3: G is regular, 2-connected, and has a depth-first tree which is not a path.



Since G is 2-connected, $G - y$ and $G - z$ are connected. DFS tree means a descendant of y is connected to an ancestor of x. Thus $G - \{y, z\}$ is connected.

Order the vertices y, z, followed by a reverse depth-first ordering of $G - \{y, z\}$ with x as the root.

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Proposition (Chartrand and Kronk)

If every DFS tree of a connected graph G is a path, then G is a cycle, a clique, or a balanced complete bipartite graph $K_{n,n}$.

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By hypothesis, G is not an odd cycle or a clique, so G must be an even cycle or a balanced complete bipartite graph on more than two vertices.

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In either case, $\chi(G) = 2 \leq \Delta(G)$. □

Greedily colour,
ensuring neighbours follow
all except the last.

Choose the last vertex wisely:
friend of few or of leaders.

Vizing's Theorem

For any simple graph G, $\chi'(G) \leq \Delta(G) + 1$.

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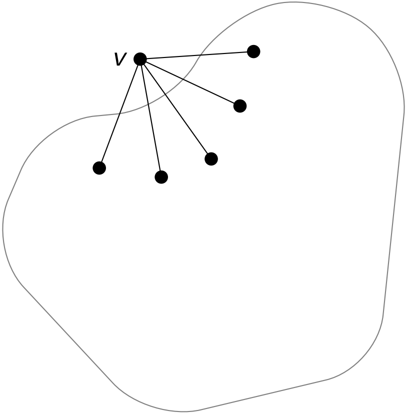
By induction. Suppose $G - v$ has a $(\Delta + 1)$-edge-colouring.



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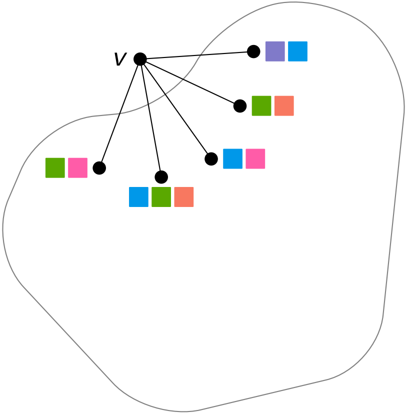

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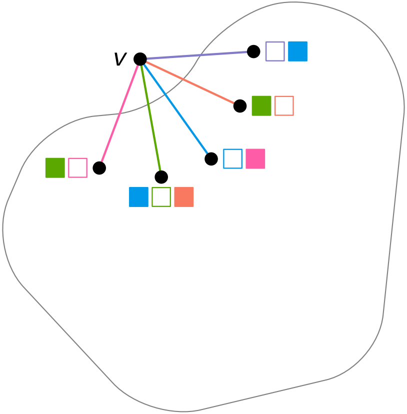
By induction. Suppose $G - v$ has a $(\Delta + 1)$-edge-colouring.

Consider the colours available at each neighbour of v. If we can find an SDR, we're done.



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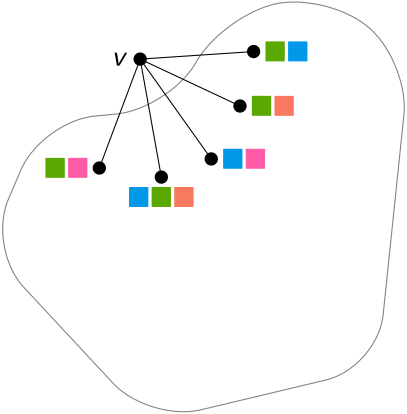

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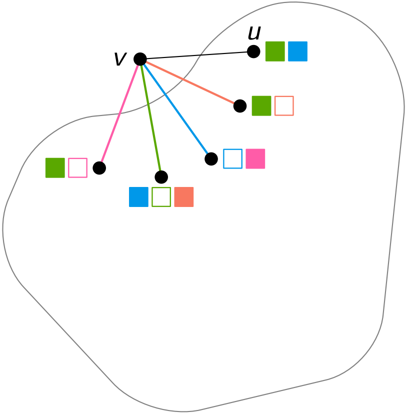

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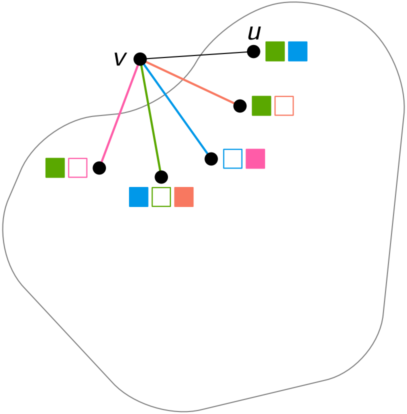
Colour α is still available at two vertices, but not at v.



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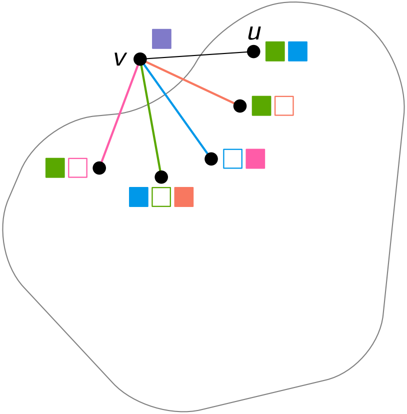
Colour α is still available at two vertices, but not at v.

Colour β is available at v, but none of its neighbours.



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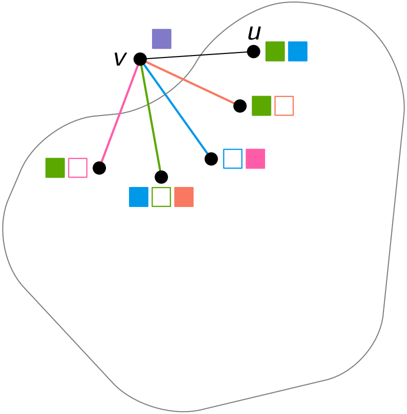
Idea: swap β for α at one vertex, then make a trade to colour vu.



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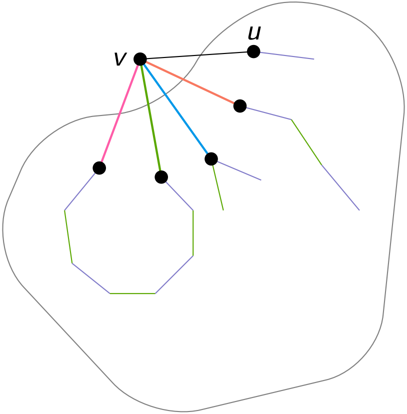

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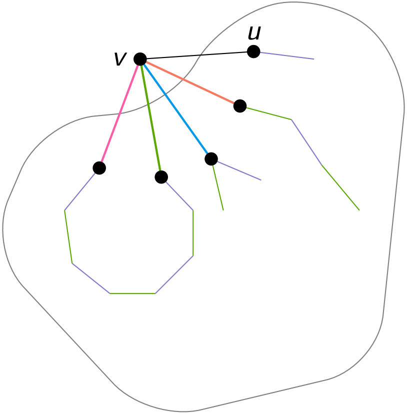

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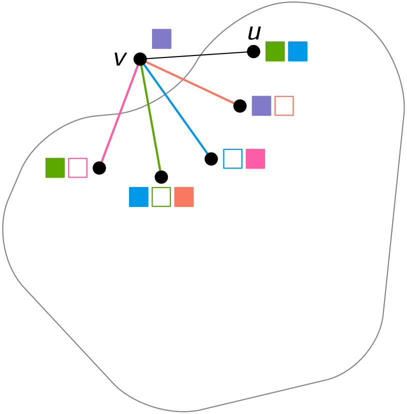

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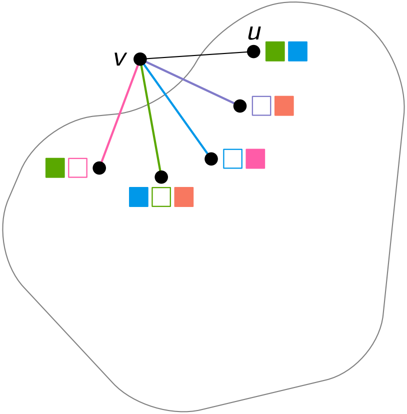
Colour α is still available at two vertices, but not at v.

Colour β is available at v, but none of its neighbours.

Idea: swap β for α at one vertex, then make a trade to colour vu.



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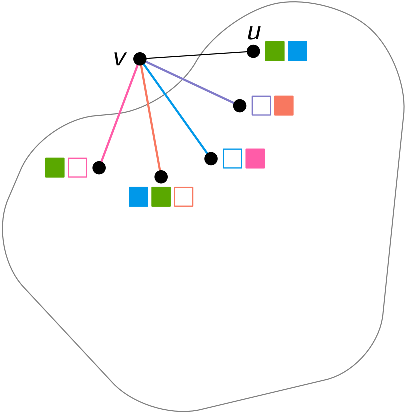

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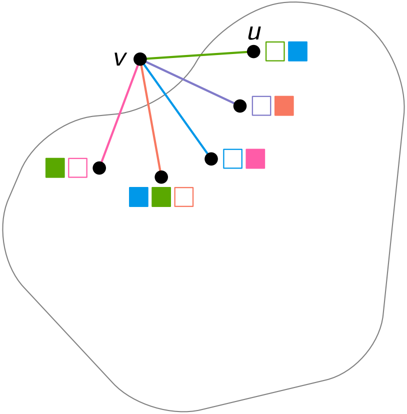

Induction on n.
Swap available colours
and find SDR.

Graph theory haiku

Three short and beautiful proofs

Ross Churchley

University of Victoria

December 21, 2021